ISSN 1064-5624, Doklady Mathematics, 2013, Vol. 87, No. 2, pp. 193–197. © Pleiades Publishing, Ltd., 2013.
Original Russian Text © I.V. Melnikova, U.A. Alekseeva, 2013, published in Doklady Akademii Nauk, 2013, Vol. 449, No. 4, pp. 393–397.

MATHEMATICS

Solution of an Abstract Cauchy Problem with Nonlinear and Random Perturbations in the Colombeau Algebra

I. V. Melnikova and U. A. Alekseeva

Presented by Academician A.N. Shiryaev October 23, 2012

Received August 17, 2012

DOI: 10.1134/S106456241302021X

INTRODUCTION

This paper solves the abstract quasilinear stochastic Cauchy problem

$$X'(t) = AX(t) + F(X(t)) + BW(t), \quad t \geq 0, \quad X(0) = f, \tag{1}$$

where A is the generator of a C_0 semigroup of operators or an integrated semigroup on a Hilbert space H, F is a nonlinear mapping of H, $W = \{W(t), t \geq 0\}$ is a white noise process with values in a Hilbert space $\mathbb{H}$, and $B \in \mathcal{L}(\mathbb{H}, H_a)$, where H_a is an algebra in H.

Problem (1) arises in many evolution models taking into account nonlinear and random perturbations. Its difficulty is caused by three reasons. The first reason is the irregularity of the white noise process, which is a consequence of its infinite variation, the independence of the random variables $W(t_1)$ and $W(t_2)$ for any $t_1 \neq t_2$, and the infinite dimension of the space $\mathbb{H}$; the second is the nonlinear component on the Hilbert space; and, finally, in the case of integrated semigroups, the operators of the solution of the corresponding homogeneous problem are unbounded. All this requires seeking appropriate approaches to the interpretation of the white noise and nonlinear terms, to setting the stochastic problem involving them, and to the construction of a solution.

At present, the best known approach to solving equations with white noise consists in passing from differential problems to the corresponding integral problems and interpreting the white noise integral as an Ito integral over a Wiener process (see, e.g., [3] or, in the numerical case, [9]). Another way to overcome the irregularity of white noise consists in studying problem (1) in spaces of abstract distributions. For linear problems (with $F = 0$), in this setting, both the problem of determining white noise and the problem related to the unboundedness of the solution operators of the homogeneous problem have been overcome; namely, solutions generalized in the time variable, the random variable, and the variable of the space H have been constructed [2, 5]. However, for the nonlinear problem (1), such an approach involves multiplying generalized functions, which are, in addition, Hilbert-valued. This requires a new technique for handling both the irregularity of white noise and multiplication of generalized functions.

This paper suggests an adaptation of Colombeau's approach to multiplying generalized functions for the abstract case in the context of the setting of problem (1) and constructing generalized solutions of this problem.

1. DEFINITION OF THE COLOMBEAU ALGEBRA OF ABSTRACT STOCHASTIC PROCESSES

The space of abstract (Hilbert-valued) Colombeau generalized functions is defined as follows. Let $\mathcal{A}_0$ denote the set of functions from $\mathcal{D}$ satisfying the condition $\int \varphi(t)dt = 1$, and let $\mathcal{A}_q$ $(q \in \mathbb{N})$, be the set of $\varphi \in \mathcal{A}_0$ satisfying the condition $\int t^k \varphi(t)dt = 0$, $k = 1, 2, \ldots, q$. Obviously, the set $\mathcal{A}_1$ is nonempty, because it contains all even functions from $\mathcal{A}_0$; the nonemptiness of $\mathcal{A}_2$ and all $\mathcal{A}_q$ with $q > 2$ can be proved by directly constructing elements of these spaces [7].

Let H_a be an algebra in the Hilbert space H, e.g., the set of continuous or finitely differentiable functions from the space $L_2(\mathbb{R})$ endowed with the corresponding norm, which turns this set into a Banach space. Consider the space

Ural Federal University,
pr. Lenina 51, Yekaterinburg, 620083 Russia
e-mail: irina.melnikova@usu.ru, uliana.alekseeva@usu.ru

$$\mathscr{E}(H_a) := (C^{\infty}(\mathbb{R}; H_a))^{\mathscr{A}_0}$$
$$= \{u: \mathscr{A}_0 \to C^{\infty}(\mathbb{R}; H_a)\}.$$

The elements of this space can be treated as H_a-valued functions of two arguments, $\varphi \in \mathscr{A}_0$ and $t \in \mathbb{R}$, i.e., as $u: \mathscr{A}_0 \times \mathbb{R} \to H_a$, $u = u(\varphi)(t) = u(\varphi, t)$; the first argument serves as a parameter (here, the functions φ do not play the role of test functions, which they play in the construction of spaces of generalized functions), and with respect to the second argument, the mapping is infinitely differentiable. For the elements of the space $\mathscr{E}(H_a)$ thus introduced, multiplication and differentiation are defined as

$$(uv)(\varphi) := u(\varphi)v(\varphi),$$
$$u^{(n)}(\varphi) := \frac{d^n}{dt^n}u(\varphi), \quad \varphi \in \mathscr{A}_0,$$

so that $\mathscr{E}(H_a)$ becomes an associative commutative differential algebra.

Let $\mathscr{D}'(H_a)$ denote the subset of distributions in $\mathscr{D}'(H)$ with values in H_a; this set is embedded in $\mathscr{E}(H_a)$ by means of the mappings

$$i: \mathscr{D}'(H_a) \to \mathscr{E}(H_a), \quad (iu)(\varphi) := u * \varphi,$$
$$u \in \mathscr{D}'(H_a), \quad \varphi \in \mathscr{A}_0.$$

This is indeed an embedding, because the mapping i is injective, which follows from the relation $\lim_{\varepsilon \to 0}(u * \varphi_\varepsilon) = u$, where $\varphi_\varepsilon(t) := \varepsilon^{-1}\varphi(t\varepsilon^{-1})$, for $\varepsilon > 0$ and $\varphi \in \mathscr{A}_0$. In constructing a differential algebra containing distributions and consisting of infinitely differentiable functions (for each fixed φ), it is natural to expect that the mapping i takes the product of infinitely differentiable functions of t to the product of their images as elements of the algebra. In order to "calibrate" the space $\mathscr{E}(H_a)$ so as to satisfy this condition, we perform the following construction. We define the space $\mathscr{E}_M(H_a)$ (M is for "moderate") consisting of those elements $u \in \mathscr{E}(H_a)$ which satisfy the condition

(M) for any compact set $K \subset \mathbb{R}$ and any $n \in \mathbb{N}_0$, there exists a $q \in \mathbb{N}$ such that

$$\forall \varphi \in \mathscr{A}_q \ \exists C > 0, \quad \delta > 0: \ \sup_{t \in K}\left\|\frac{d^n}{dt^n}u(\varphi_\varepsilon)(t)\right\|_H \le \frac{C}{\varepsilon^q},$$
$$0 < \varepsilon < \delta.$$

Additionally, we introduce the null space $\mathscr{N}(H_a)$ consisting of those elements $u \in \mathscr{E}(H_a)$ which satisfy the condition

(N) for any compact set $K \in \mathbb{R}$ and any $n \in \mathbb{N}_0$, there exists a $q \in \mathbb{N}$ such that

$$\forall \varphi \in \mathscr{A}_p \ \exists C > 0, \ \delta > 0: \ \sup_{t \in K}\left\|\frac{d^n}{dt^n}u(\varphi_\varepsilon)(t)\right\|_H \le C\varepsilon^{p-q},$$
$$0 < \varepsilon < \delta, \quad p \ge q.$$

The elements of the space $\mathscr{E}_M(H_a)$ form a differential algebra, and $\mathscr{N}(H_a)$ is a differential ideal in this algebra. Consider the quotient algebra

$$\mathscr{G}(H_a) := \mathscr{E}_M(H_a)/\mathscr{N}(H_a).$$

The algebra $\mathscr{G}(H_a)$, as well as $\mathscr{G}(\mathbb{R})$, is an associative commutative H_a-valued differential algebra. Moreover, since all images $i(u)$ of distributions $u \in \mathscr{D}'(H_a)$ fall in $\mathscr{E}_M(H_a)$ (by virtue of the structure theorem for $\mathscr{D}'(H_a)$) and the preimage of the null space $\mathscr{N}(H_a)$ under i is the zero element of the space $\mathscr{D}'(H_a)$, it follows that the images of distributions are elements of the quotient algebra defined above. We denote the elements of the quotient algebra $\mathscr{G}(H_a)$ (that is, classes of mappings) by capitals U, V, ... and representatives of classes, by the corresponding small letters.

The support of an element $U \in \mathscr{G}(H_a)$ is defined as follows. Let Λ be an open subset of $\mathbb{R}$, and let $\mathscr{G}_\Lambda(H_a)$ be the differential quotient algebra whose elements are classes of mappings from $\mathscr{A}_0$ to $C^\infty(\Lambda, H_a)$ satisfying the corresponding conditions (M) and (N). If the restriction of U to Λ vanishes in $\mathscr{G}_\Lambda(H_a)$, then we say that U vanishes on Λ. The complement to the maximal set Λ on which U vanishes is called the support of the element U. If $u \in \mathscr{D}'(H_a)$, then, as in the numerical case [7, 8], the support of the image $i(u)$ in the space $\mathscr{G}(H_a)$ coincides with the support of u in $\mathscr{D}'(H_a)$.

Now, we define the algebra $\mathscr{G}(\Omega, H_a)$ of $\mathscr{G}(H_a)$-valued random processes $U = \{U(\omega), \omega \in (\Omega, \mathscr{B}(\Omega), \mu)\}$ as measurable mappings from $(\Omega, \mathscr{B}(\Omega), \mu)$ to $\mathscr{G}(H_a)$. This means that, for any $\varphi \in \mathscr{A}_0$, at fixed t, the preimage of any Borel set from H_a under the mapping $U(\varphi, \omega)$ (by any representative of the class U) belongs to $\mathscr{B}(\Omega)$.

2. DEFINITION OF THE WHITE NOISE PROCESS AS AN ELEMENT OF THE STOCHASTIC ALGEBRA

We define the white noise process W as an ω-measurable generalized (in t) function with values in $\mathbb{H}$. One of the ways of such a definition is based on ideas of abstract stochastic distributions (see, e.g., [5, 2]). Let $\mathscr{S} = \mathscr{S}(\mathbb{R})$ be the space of rapidly decreasing functions. By $\mathscr{S}'(\mathbb{H})$ we denote the space of $\mathbb{H}$-valued distributions over $\mathscr{S}$. According to the generalization of the Bochner–Minlos theorem to the case of Hilbert space valued generalized functions [1], there exists a unique probability measure μ on the Borel σ-algebra $\mathscr{B}(\Omega)$ generated by the weak topology of the space $\Omega := \mathscr{S}'(\mathbb{H})$ and a trace operator Q satisfying the condition

$$\int_\Omega e^{i(\langle\omega, \theta\rangle, h)_{\mathbb{H}}} d\mu(\omega) = e^{-\frac{1}{2}\|\theta\|^2(Qh, h)},$$
$$\theta \in \mathscr{S}, \quad h \in \mathbb{H}, \quad \|\theta\| = \|\theta\|_{L_2(\mathbb{R})}.$$

This allows us to define, by using the identity mapping, the white noise process on the probability space

$(\Omega, \mathcal{B}(\Omega), \mu)$ as a generalized (in t) function from $\mathcal{D}'(\mathbb{H}) \supset \mathcal{S}'(\mathbb{H})$:

$$\langle W, \theta\rangle = \langle W(\cdot, \omega), \theta(\cdot)\rangle := \langle \omega, \theta\rangle, \quad \theta \in \mathcal{D}.$$

This process is a generalization of the corresponding $\mathbb{R}$-valued Gaussian process; its mathematical expectation vanishes, and $\mathrm{Cov}\langle W, \theta\rangle = \|\theta\|^2 Q$. Bearing in mind our purposes in this paper, which are setting and solving the Cauchy problem (1), we set W to zero on $(-\infty, 0)$; for the resulting process supported on $[0, \infty)$, we use the same notation:

$$\langle W(\cdot, \omega), \theta(\cdot)\rangle := (-1)^k \langle (\chi \cdot W^{(-k)})(\cdot), \theta^{(k)}(\cdot)\rangle, \quad \theta \in \mathcal{D}.$$

Here, $W^{(-k)}$ is a continuous kth-order antiderivative of the generalized function W, whose existence is guaranteed by the structure theorem, and χ is the Heaviside function.

Another method for defining a generalized $\mathbb{H}$-valued white noise process on an arbitrary probability space $(\Omega, \mathcal{B}(\Omega), \mu)$ is to differentiate the Hilbert space valued ($\mathbb{H}$-valued in the case under consideration) Q-Wiener process $\{\mathcal{W}_Q(t), t \ge 0\}$ [3, 2] extended by zero over $(-\infty, 0)$, i.e., to set $\langle W, \theta\rangle := -\langle \mathcal{W}_Q, \theta'\rangle$ for $\theta \in \mathcal{D}$.

To complete setting the problem, we embed the constructed generalized white noise process in the Colombeau algebra $\mathcal{G}(\Omega, H_a)$. The convolution of the generalized function $W(\cdot, \omega) \in \mathcal{D}'_+(\mathbb{H})$ with a function from $\mathcal{A}_0$ is the function

$$w(\varphi, t, \omega) := \langle W(\cdot, \omega), \varphi(t - \cdot)\rangle, \quad \varphi \in \mathcal{A}_0, \quad t \in \mathbb{R}, \tag{2}$$

which is infinitely differentiable in $t \in \mathbb{R}$ (and, as previously, measurable in $\omega \in \Omega$). Thus, $w(\varphi, \cdot, \omega) \in C^\infty(\mathbb{R}, \mathbb{H})$, $\omega \in \Omega$ almost surely, and $w(\varphi, \cdot, \cdot) \in C^\infty(\mathbb{R}, L_2(\Omega, \mathbb{H}))$. Applying the operator $B \in \mathcal{L}(\mathbb{H}, H_a)$ to w, we obtain $Bw(\varphi, t, \omega) \in H_a$. Therefore, $Bw(\varphi, \cdot, \cdot)$: $\mathcal{A}_0 \to C^\infty(\mathbb{R}, L_2(\Omega, H_a))$ is a representative of some class in the algebra $\mathcal{G}(\Omega, H_a)$. We denote the corresponding class by BW; it is this sense in which the stochastic term in Eq. (1) is understood.

3. SOLUTION OF EQUATIONS WITH INFINITELY DIFFERENTIABLE NONLINEARITY F

Let us make several assumptions. We set $H = \mathbb{H} = L_2(\mathbb{R})$ and let F be an infinitely differentiable function of a real variable bounded on $\mathbb{R}$ together with all of its derivatives and such that $F(0) = 0$. Suppose that the domain of the operator A is contained in the set of continuous functions in $L_2(\mathbb{R})$. Then, on $L_2(\mathbb{R}) \cap \mathrm{dom}A$, the product of elements of the space $L_2(\mathbb{R})$ is determined as the pointwise product of continuous functions. For H_a we take the set of m times differentiable functions in $L_2(\mathbb{R})$ and assume that $H_a \subset \mathrm{dom}A$. Such a situation is typical for, e.g., differential operators A. We first consider the case where the operator A generates a strongly continuous C_0 semigroup $\{S(t), t \ge 0\}$ in $L_2(\mathbb{R})$.

For problem (1) with these A and F and with stochastic term BW defined at the end of the preceding section, we seek solutions belonging to the stochastic abstract Colombeau algebra $\mathcal{G}(\Omega, H_a)$. By the definition of the space H_a, for the operator $B \in \mathcal{L}(H, H_a)$ we can take, e.g., the operator of convolution with some m times differentiable function.

We begin with answering the question on the existence of a solution to the problem

$$Y' = AY + F(Y) + BW, \quad \mathrm{supp}\, Y \subseteq [0, \infty) \tag{3}$$

as an element of the algebra $\mathcal{G}(\Omega, H_a)$. Take any $\eta > 0$ and consider a representative of the stochastic process $BW \in \mathcal{G}(\Omega, H_a)$, namely, the function $Bw(t) := Bw(\varphi, t, \omega)$, where $\varphi \in \mathcal{A}_0$ and $\omega \in \Omega$, supported on $[-\eta, \infty)$. By the definition of the elements of $\mathcal{G}(\Omega, H_a)$, for each fixed $\varphi \in \mathcal{A}_0$, $Bw(t)$ is an infinitely differentiable function of a variable $t \in \mathbb{R}$ taking values in H_a and measurable in $\omega \in \Omega$. Now, for an arbitrary function $\varphi \in \mathcal{A}_0$, consider the problem

$$y'(t) = Ay(t) + F(y(t)) + Bw(t), \quad t \ge -\eta, \qquad y(t) = 0, \quad t \le -\eta, \tag{4}$$

where $y(t) = y(\varphi, t, \omega)$, and $\varphi \in \mathcal{A}_0$, $\omega \in \Omega$. We seek a solution of this problem which belongs to $C^\infty([-\eta; \infty); H_a)$ almost surely.

As is known, in the case of a differentiable inhomogeneity, a solution of an inhomogeneous abstract Cauchy problem with operator A generating a C_0 semigroup $\{S(t), t \ge 0\}$ exists and is determined by the Cauchy formula as the convolution of the semigroup with the inhomogeneity:

$$y(t) = \int_{-\eta}^{t} S(t-s)F(y(s))ds + \int_{-\eta}^{t} S(t-s)Bw(s)ds := \mathcal{Q}y(t), \quad t \ge -\eta. \tag{5}$$

Thus, if a solution of the given equation exists and belongs to $C^\infty([-\eta; \infty); H_a)$, then this solution is also a solution of (4). The operator $\mathcal{Q}$ of problem (5) is a Volterra-type operator. Since F is differentiable and its derivatives are bounded, it follows that, on each interval $[-\eta; T]$, some power $\mathcal{Q}^k$ $(k = k(T))$ of this operator is contractive, and the sequence of approximations $y_n = \mathcal{Q}^{nk} y_0(t)$ has a unique limit $y(t) = \lim_{n\to\infty} \mathcal{Q}^{nk} y_0(t)$ in H, which is uniform in t on the interval $[-\eta; T]$. Note that, if the first point for the sequence of approximations is an infinitely differentiable (with respect to t) function $y_0(\cdot)$ with values in H_a, then the first iteration (the function $y_1(t) = \mathcal{Q}^k y_0(t)$) obtained by the k-fold superposition of the operations of the application of the infinitely differentiable function F, the semigroup operators, integration, and the addition of infinitely differentiable terms of the form $\int_{-\eta}^{t} S(t-s)Bw(s)ds$ is again infinitely differentiable (with respect to t) and takes values in H_a, as well as all of the subsequent iterations $y_n(\cdot)$.

Thus, $y_n(t) \in H_a$, but $\lim_{n\to\infty} y_n(t) = y(t)$ does not generally belong to H_a, because the algebra H_a is not closed as a subspace in H. If $y(t) \in H_a$, then Eq. (5) implies $y(t) = 0$ for $t \le -\eta$; i.e., the support of the constructed solution is contained in $[-\eta; \infty)$. It follows from the differentiability of F, the boundedness of F', and the condition $F(0) = 0$ that

$$\max_{t\in K}\|y(\varphi_\varepsilon, t, \omega)\| \le C_1\max_{t\in K}\|Bw(\varphi_\varepsilon, t, \omega)\| + C_2\max_{t\in K}\int_{-\eta}^{t}\|y(\varphi_\varepsilon, s, \omega)\|ds.$$

According to condition (M), for all $\varphi \in \mathcal{A}_q$ (with some fixed $q \in \mathbb{N}$), the first term on the right-hand side of the inequality grows no faster than ε^{-q} as $\varepsilon \to 0$. By virtue of the Gronwall–Bellman inequality, the left-hand side behaves in a similar way, which implies that the function $y(\cdot)$ satisfies condition (M) with $n = 0$. (According to the Gronwall–Bellman inequality, if $y(t) \le c + \int_{t_0}^{t} f(s)y(s)ds$, where $c > 0$ and $t > t_0$, for positive continuous functions y and f, then $y(t) \le c\exp\left(\int_{t_0}^{t} f(s)ds\right)$.) The behavior of the derivatives of $y(\cdot)$ is verified in a similar way by using the infinite differentiability of F and the boundedness of its derivatives.

Thus, y is a representative of some class $Y \in \mathcal{G}(\Omega, H_a)$. Using the Gronwall–Bellman inequality and considering the difference $y_{\eta_1} - y_{\eta_2}$, we can show that the support of Y does not depend on η and is contained in $[0; \infty)$. The uniqueness of the solution is proved in a similar way.

Since the linear Cauchy problem corresponding to (1) in the space of distributions has the form

$$\langle X', \varphi\rangle = \langle\delta, \varphi\rangle f + \langle AX, \varphi\rangle + \langle BW, \varphi\rangle, \quad \varphi \in \mathcal{D}, \tag{6}$$

it follows that the solution of the Cauchy problem (1) in $\mathcal{G}(\Omega, H_a)$ is obtained by augmenting Y by the convolution of the semigroup with the image in $\mathcal{G}(\Omega, H_a)$ of the element $\delta \cdot f$, which reflects the influence of the initial condition: $X = Y + (S * i\delta)f$ for any $f \in H_a$.

Thus, we have proved the following theorem.

Theorem. *Suppose that A is the generator of a C_0 semigroup $\{S(t), t \ge 0\}$ of operators on the space $L_2(\mathbb{R})$, F is an infinitely differentiable function on $\mathbb{R}$ bounded together with all of its derivatives such that $F(0) = 0$, $B \in \mathcal{L}(L_2(\mathbb{R}), H_a)$, the stochastic process BW is an element of the algebra $\mathcal{G}(\Omega, H_a)$, and Bw is a representative of BW supported on $[-\eta; \infty)$. Then, for any $\varphi \in \mathcal{A}_0$, Eq.* (4) *has a unique solution $y \in C^\infty([-\eta; \infty); H)$. If $y \in C^\infty([-\eta; \infty); H_a)$, then Eq.* (3) *has a unique solution in the algebra $\mathcal{G}(\Omega, H_a)$. In this case, the solution of the Cauchy problem* (1) *in $\mathcal{G}(\Omega, H_a)$ is $X(t) = Y(t) + (S * i\delta)(t)f$, where $t \ge 0$ and $f \in H_a$.*

Note that, in the general case, it follows from $\lim_{n\to\infty} y_n(t) = y(t) \notin H_a$ that we obtain only the approximate solutions of problem (5) defined by $y_n(t) = \mathcal{Q}^k y_{n-1}(t)$ for $t \ge -\eta$.

4. A REMARK ON OPERATORS GENERATING INTEGRATED SEMIGROUPS

If the operator A generates an exponentially bounded n-times integrated semigroup $\{S_n(t), t \ge 0\}$ more general than a semigroup of class C_0, then, instead of (5), the following equation holds:

$$y(\varphi, t, \omega) = \int_{-\eta}^{t} S_n(t-s)F^{(n)}(y(\varphi, s, \omega))ds + \int_{-\eta}^{t} S_n(t-s)(Bw)^{(n)}(\varphi, s, \omega)ds,$$
$$\varphi \in \mathcal{A}_0, \quad \omega \in \Omega, \quad t \ge -\eta.$$

Here, all derivatives with respect to t exist by virtue of the infinite differentiability of F and Bw. In this case, as well as in the case of a class C_0 semigroup, the question about the solvability of Eq. (3) and the Cauchy problem (1) in the space $\mathcal{G}(\Omega; H_a)$ reduces to the question of whether the limit of the sequence of approximations y_n exists and belongs to H_a.

Examples of operators A satisfying the conditions specified above and generating different integrated semigroups are various differential operators $A = A\left(i\frac{d}{dx}\right)$ of systems well-posed in the sense of Petrovskii [4, 6]. Such operators can be perturbed by bounded operators of any nature, still remaining the generators of integrated semigroups.

ACKNOWLEDGMENTS

This work was supported by the Ministry of Education and Science of the Russian Federation (program no. 1.1016.2011) and by the Russian Foundation for Basic Research (project no. 10-01-96003r).

REFERENCES

1. M. A. Alshanskiy, Russian Math. (Izv. VUZ. Mat.) **48** (2), 8–15 (2004).
2. M. A. Alshanskiy and I. V. Melnikova, Sb.: Math. **202** (11), 1565–1592 (2011).
3. G. da Prato, *Kolmogorov Equations for Stochastic PDEs* (Birkhauser, Basel, 2004).
4. I. V. Melnikova and A. Filinkov, *The Cauchy Problem: Three Approaches* (Chapman & Hall, New York, 2001).
5. I. V. Melnikova, A. I. Filinkov, and M. A. Alshanskiy, J. Math. Sci. **116** (5), 3620–3656 (2003).
6. I. V. Melnikova and U. A. Anufrieva, J. Math. Sci. **148** (4), 481–632 (2008).
7. M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations* (Longman, Harlow, 1992).
8. M. Oberguggenberger and F. Russo, in *Stochastic Analysis and Related Topics, IV*, Ed. by L. Decreusefond, J. Gjerde, B. Öksendal, and A. S. Ustünel (Birkhäuser, Boston, 1998), pp. 319–322.
9. B. Öksendal, *Stochastic Differential Equations* (Springer, Berlin, 2000).

Translated by O. Sipacheva